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Abstract. Group cohomology techniques are used to derive conditions for a group $\tilde{G}$ to be a local splitting group for G. As a first step the exactness of the inflation-restriction sequence gives a characterisation of factor systems arising in locally operating representations of transitive transformation groups. Some examples and applications of the theory are also given.

1. Introduction

Kinematic groups of space–time transformations have played a very important role in the conceptual development of quantum theory. For instance, Poincaré and Galilei invariances and their relationship to the concept of elementary particle are well known by most physicists. More recently, Hoogland (1976a, b, 1977) has shown that the classification of elementary systems according to the equivalence classes of (semi-unitary) projective representations of the kinematic group cannot be considered as satisfactory and he pointed out that the representations of kinematic groups which are relevant for quantum mechanics are those christened by him as 'locally operating representations'. This concept however was roughly defined and it seemed to be lacking a more precise definition. An intrinsic geometric formulation of such representations in the case of linear representations was recently given by Asorey *et al* (1983). A sketch of how to deal with the more general case of multiplier representations was also presented in a paper by Cariñena *et al* (1982), where the possibility was shown of reducing the problem to the older one of linear representations, but with the replacement of the group G by a new group $\tilde{G}$. The case of Galilei group and Galilean relativistic wave equations was studied by Cariñena and Santander (1982).

The aim of this paper is to look at some mathematical points which have not been considered till now, as far as we know. They are the characterisation of factor systems arising in locally operating representations and the identification of candidates for local splitting and representation groups. The tool is the inflation-restriction sequence (Cattaneo 1978), also called exact homology sequence (MacLane 1975), such as it was recently indicated by Cariñena *et al* (1983).

The determination of a splitting group of G has been shown to be very adequate for the physical interpretation of some parameters characterising the cocycles defining

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the different extensions of G (see e.g. Cariñena and Santander 1975). It is well known since Bargmann's classical work (1954) that any projective representation P of a group G can be lifted to a linear representation of a group which is an extension of G by T but the point is that such a group depends on P. On the contrary, one splitting group $\tilde{G}$ is enough for the lifting of any projective representation of G. The values of the parameters characterising the cocycles arise now as characterising the different representations of $\tilde{G}$ and therefore are on the same footing as the remaining kinematic observables.

In order for this paper to be self contained, we give in § 2 a short summary of the inflation-restriction sequence, in the case we are considering in which A and T are trivial Polish G-modules. In § 3 we will make use of this sequence to characterise the set of factor systems of locally operating representations of a transitive group of transformations. The existence and characterisation of local splitting and representation groups, reducing the problem of multiplier locally operating representations of G to that of linear locally operating representations of such splitting groups, is analysed in § 4. Finally, in § 5 we show by means of some examples how the theory we have developed works.

2. The inflation-restriction sequence

In this section we will summarise some ideas about the inflation-restriction sequence in the simpler case we are going to use in the next section. For more detailed information we refer to appendix B of Cattaneo's paper (1978). Let (E, ρ) be a topological central extension of the Polish connected group G by the Abelian Polish group A. This means that the sequence $1 \to A \xrightarrow{i} E \xrightarrow{\rho} G \to 1$ is exact, ρ is a continuous epimorphism and $i: A \to i(A) \subset E$ is an injective homomorphism. If T is the circle group, the inflation inf^n $(n = 1, 2)$, restriction, res^1, and transgression map, trg^1, are the maps defined as follows. The map $\mathrm{inf}^n: H^n(G, T) \to H^n(E, T)$ $(n = 1, 2)$ is that induced from $f \to f \circ \rho^n$ of $Z^n(G, T)$ into $Z^n(E, T)$ which maps $B^n(G, T)$ into $B^n(E, T)$. In a similar way is defined $\mathrm{res}^1: H^1(E, T) \to H^1(A, T)$. On the other hand, $\mathrm{trg}^1: H^1(A, T) \to H^2(G, T)$ is defined by making use of the factor system $W_\sigma: G \times G \to A$ associated to a (Borel) section $\sigma: G \to E$ by passing to the quotients the map of $Z^1(A, T)$ in $Z^2(G, T)$, $\chi \to \chi \circ W_\sigma$. The cohomology groups we are considering are those of the so called Mackey-Moore cohomology (see Cattaneo and Janner 1974 and Moore 1964), the action of G on T being trivial. The fundamental point is that the following inflation-restriction sequence for (E, G, ρ, T)

$$1 \to H^1(G, T) \xrightarrow{\mathrm{inf}^1} H^1(E, T) \xrightarrow{\mathrm{res}^1} H^1(A, T) \xrightarrow{\mathrm{trg}^1} H^2(G, T) \xrightarrow{\mathrm{inf}^2} H^2(E, T)$$

is exact. A proof of this fact can be found in Cattaneo (1978). It is noteworthy that for Polish groups, according to a theorem by Banach (1931), Borel 1-cocycles are continuous and consequently the inflation-restriction sequence can also be written as follows

$$1 \to \hat{G} \to \hat{E} \to \hat{A} \to H^2(G, T) \to H^2(E, T),$$

where the caret over the capital letters denotes the set of one-dimensional unitary continuous representations of the corresponding group, endowed with the compact-open topology relative to the original one.

3. Characterisation of factor systems of locally operating representations

Let G be a connected Polish group of transformations of a topological space M. The action of G on M is assumed to be transitive and if x_0 is an arbitrary but fixed point in M, the space M may be identified to G/Γ, where Γ is the isotopy group of x_0. By a locally operating multiplier representation of G we mean a Borel multiplier representation of G in which the representation space is made up by vector valued functions $F: M \to \mathbb{C}^m$ and the representation of G is given by $[U(g)F](gx) = A(g,x)F(x)$, where $A: G \times M \to \mathrm{GL}(M;\mathbb{C})$ is a matrix function satisfying $A(g_2, g_1x)\, A(g_1, x) = \omega(g_2, g_1)A(g_2g_1, x)$ which is called a gauge matrix. The function $\omega: G \times G \to T$, $\omega \in Z^2(G, T)$, is the factor system of the representation U.

The problem of (linear) locally operating representations has recently been studied on a paper by Asorey *et al* (1983) and more information, both from the mathematical and physical viewpoints can be found in the excellent lectures of Isham (1983), especially §§ 1 and 5, where it is shown that the definition given above for $U(g)$ is the local expression of a 'twisted' representation, i.e. $U(g)$ corresponds to a transformation in the space of sections of a vector bundle with base M. Locality now means that the support of the section $U(g)\psi$ is contained in the image of the support of ψ. From the infinitesimal viewpoint, the generators will be represented by local operators (which decrease supports) and it follows from a theorem of Peetre (1960) that they will be differential operators. In this paper we will deal with multiplier representations instead of linear ones and therefore factor systems ω can appear and are going to be studied. In this section we will be interested in the characterisation of factor system which can arise in locally operating representations. The particular case of A being one dimensional and that of Γ being a connected and simply connected group can be easily answered. In fact, the defining relation for ω given above, when restricted to $\Gamma \times \{x_0\}$, leads to a multiplier representation of Γ, namely $A(\gamma', x_0).\ A(\gamma, x_0) = \omega(\gamma', \gamma)A(\gamma'\gamma, x_0)$, with factor system $\tau = \omega_{|\Gamma\times\Gamma}$. In the above mentioned cases such systems must be trivial (Bargmann 1954) and consequently only factor systems ω whose restrictions to Γ are equivalent to the trivial one can arise in a multiplier locally operating representation of such groups.

The less trivial and more general case of Γ being a connected, but possibly not simply connected, Lie group can be analysed by using the inflation-restriction sequence corresponding to the covering homomorphism $\rho: \Gamma^* \to \Gamma$, Γ^* being the universal covering group of Γ, i.e., the short exact sequence $1 \to \pi_1(\Gamma) \to \Gamma^* \to \Gamma \to 1$. We will denote by α, Λ and δ respectively the restriction, inflation and transgression homomorphisms. The first homotopy group is injected in the centre of Γ^* and consequently is a trivial Γ-module. In the case of Γ being connected, in which we are interested, the action of Γ on the circle group to be considered is the trivial action. Consequently, the inflation-restriction sequence is but $1 \to \hat{\Gamma} \to \hat{\Gamma}^* \xrightarrow{\alpha} \widehat{\pi_1(\Gamma)} \xrightarrow{\delta} H^2(\Gamma, T) \xrightarrow{\Lambda} H^2(\Gamma^*, T)$ and exactness of this sequence at $H^2(\Gamma, T)$ means that $\delta(\widehat{\pi_1(\Gamma)}) = \ker \Lambda$. The important point is that any finite-dimensional projective $\bar{\tau}$-representation of Γ, P, gives rise to a projective representation $P \circ p$ of Γ^* with factor system $\Lambda(\bar{\tau})$ which has to be trivial because of the simply connectedness of Γ^*. Consequently, the restriction to $\Gamma \times \Gamma$ of factor systems $\bar{\omega}$ of locally operating multiplier representations of G will be in the kernel of the inflation map.

Hereafter we will be restricted to the very general case where Γ is such that for every $\bar{\tau}$ in the kernel of the inflation map there exists a finite-dimensional τ-representation of Γ. It is then possible to show that the set $H^2_{\mathrm{loc}}(G, T)$ of factor systems of locally operating representations is just that of factor systems such that their restrictions to $\Gamma\times\Gamma$ are in the kernel of the inflation map. As a first remark it is worthwhile noting that for any $\bar\omega\in H^2(G,T)$ and any arbitrary Borel section $s:M\to G$ we can find a lift $\omega\in Z^2(G,T)$ such that $\omega_{|s(M)\times\Gamma}=1$. Actually, if ω' is a lifting of $\bar\omega$ and $\mu:G\to T$ is defined by $\mu(g)=\omega'(s(gx_0),\gamma(g))$, where $\gamma(g)=s(gx_0)^{-1}g\in\Gamma$, one easily checks that $\omega=\omega'\delta\mu$ satisfies $\omega_{|s(M)\times\Gamma}=1$. Moreover, $\omega_{|\Gamma\times\Gamma}=\omega'_{|\Gamma\times\Gamma}$. The next lemma will prove the identity of $\ker\Lambda$ and $H^2_{\mathrm{loc}}(G,T)$.

Lemma. Let $\bar\omega\in H^2(G,T)$ be such that its restriction to $\Gamma\times\Gamma$ is in the kernel of the inflation map, ω a lifting of $\bar\omega$ satisfying $\omega_{|s(M)\times\Gamma}=1$ and D a finite-dimensional τ-multiplier representation of Γ with $\tau=\omega_{|\Gamma\times\Gamma}$. A locally operating representation of G with factor systems in $\bar\omega$ is obtained by defining

$$[U(g)f](gx)=\omega(g,s(x))D(\gamma(gs(x)))f(x).$$

Proof. The gauge matrix A is a Borel function by construction. By taking into account identities such as $gs(x)=s(gx)\cdot\gamma(gs(x))$,

$$\gamma(g'gs(x))=\gamma(g's(gx))\gamma(gs(x))$$

and other similar ones, the cocycle identity for ω and the additional hypothesis $\omega_{|s(M)\times\Gamma}=1$, we easily check that

$$A(g_2,g_1x)A(g_1,x)A^{-1}(g_2g_1,x)=\omega(g_2,g_1)$$

which proves the result of the lemma.

As a consequence of this lemma we reach the conclusion that, in the case we are considering in which Γ satisfies the aforementioned condition, $H^2_{\mathrm{loc}}(G,T)$ is made up of all factor systems $\omega\in H^2(G,T)$ whose restrictions to $\Gamma\times\Gamma$ are in the kernel of the inflation map. As a byproduct we obtain that $H^2_{\mathrm{loc}}(G,T)$ is a closed subgroup of $H^2(G,T)$. In fact, if the restrictions to $\Gamma\times\Gamma$ of $\bar\omega_1$ and $\bar\omega_2$ are in $\ker\Lambda$, that of the product $\bar\omega_1\cdot\bar\omega_2^{-1}$ is also in $\ker\Lambda$. On the other hand, let $\{\omega_n\}_{n\in N}$ be a sequence of elements in $H^2_{\mathrm{loc}}(G,T)$ that converges to $\omega\in H^2(G,T)$. Then $\omega_{n|\Gamma\times\Gamma}$ are in $\ker\Lambda$ and as it is closed, $\omega_{|\Gamma\times\Gamma}\in\ker\Lambda$. Consequently, $H^2_{\mathrm{loc}}(G,T)$ will be closed.

4. Local representation groups

The inflation-restriction sequence can also be very useful for the characterisation of splitting and representation groups for G. We recall that if G is a Polish group and $p:\tilde G\to G$ an epimorphism, any splitting unitary representation R of $\tilde G$ (i.e. mapping $\ker p$ on T) defines a continuous unitary projective representation P of G such that $\pi\circ R=P\circ p$, where π is the canonical projection $\pi:\mathcal{U}(\mathcal{H})\to P\mathcal{U}(\mathcal{H})$. The representation R is then said to be a lifting of P to $\tilde G$. The pair $(\tilde G,p)$ is a splitting group for G if any continuous projective representations of G admits a linear lifting to $\tilde G$. We are interested in multiplier representations of G that are related to projective representations of G by means of the canonical projection π and the choice of a (Borel) section η for π in such a way that, if U is a multiplier representation, $\pi\circ U$ is a projective representation and, for any projective representation P, $\eta\circ P$ is a multiplier representation. Notice that continuous projective representations correspond to Borel multiplier representations. If the action of $\tilde{G}$ on M is defined via the projection p, the multiplier representation associated to the projective representation defined by a locally operating linear representation of $\tilde{G}$ will be a locally operating representation of G. If $(\tilde{G}, p)$ is such that any locally operating multiplier representation of G admits such a locally operating linear lifting to $\tilde{G}$, we will say that $(\tilde{G}, p)$ is a local splitting group for G. The problem is to analyse the existence of such a group and in the affirmative case its non-unicity, that is, the search for a 'minimal' one which will be called a local representation group $\bar{G}$.

As a first remark we want to point out that the same method used in Cariñena and Santander (1979) shows that in the search for local splitting groups for a connected group G it is enough to consider central extensions of G by an Abelian kernel and therefore we will consider a central extension $1 \to A \to \tilde{G} \to G \to 1$ as well as the corresponding inflation restriction sequence

$$1 \to \hat{G} \xrightarrow{\mathrm{inf}^1} \hat{\tilde{G}} \xrightarrow{\mathrm{rest}^1} \hat{A} \xrightarrow{\mathrm{trg}^1} H^2(G, T) \xrightarrow{\mathrm{inf}^2} H^2(\tilde{G}, T).$$

The factor systems of locally operating representations of G that can be lifted to linear locally operating representations of $\tilde{G}$ are those in $\mathrm{trg}^1(\hat{A})$ and therefore in order for $(\tilde{G}, p)$ to be a splitting group, $\mathrm{trg}^1(\hat{A}) \supset H^2_{\mathrm{loc}}(G, T)$. This shows that a minimality condition for $\hat{A}$ is that $\hat{A}$ be isomorphic to $H^2_{\mathrm{loc}}(G, T)$; consequently we should look for local representation groups which are central extensions of G by the dual group of $H^2_{\mathrm{loc}}(G, T)$ endowed with an appropriate topology to be determined.

As we have done in the preceding section we will only consider the case of Γ being such that for any $\bar{\sigma} \in \ker \Lambda$ there exists a finite dimensional σ-representation of Γ, i.e. $H^2_{\mathrm{loc}}(G, T) = \{\bar{\omega} \in H^2(G, T), \bar{\omega}_{\Gamma\times\Gamma} \in \ker \Lambda\}$. We analyse first the algebraic conditions determining a local representation group leaving aside the topological features. Then we can follow step by step the theory developed by Cariñena and Santander (1979) for finite groups with the only substitution of the subgroup $H^2_{\mathrm{loc}}(G, T)$ for $H^2(G, T)$ and $Z^2_{\mathrm{loc}}(G, T)$ for $Z^2(G, T)$ and we can assert that a local representation group for G determines an automorphism θ of $H^2_{\mathrm{loc}}(G, T)$ and a homomorphic section $s: H^2_{\mathrm{loc}}(G, T) \to Z^2_{\mathrm{loc}}(G, T)$. Conversely, given $\theta \in \mathrm{Aut}_{\mathrm{alg}}(H^2_{\mathrm{loc}}(G, T))$ and a homomorphic section $s: H^2_{\mathrm{loc}}(G, T) \to Z^2_{\mathrm{loc}}(G, T)$, the extension of G by $\widehat{H^2_{\mathrm{loc}}(G, T)}$ given by the factor system $W_{\theta s}(g, h): \bar{\omega} \to s[\theta^{-1}(\bar{\omega})](g, h)$ is a representation group for G. The existence of such a section s follows because $B^2_{\mathrm{loc}}(G, T)$ is divisible, like in the corollary to Theorem 5 in the paper by Cariñena and Santander (1979). Furthermore it was also proved that we can forget the automorphism θ.

As far as topological aspects are concerned we recall that, in the case of a representation group where $H^2(G, T)$ instead of $H^2_{\mathrm{loc}}(G, T)$ arises, the possibility of endowing $\bar{G}$ with a topological Polish structure depends on the existence of a locally compact topology on $H^2_{\mathrm{loc}}(G, T)$ such that for every pair (g_1, g_2) of elements in G the maps $W(g_1, g_2): \bar{\omega} \to [s(\bar{\omega})](g_1, g_2)$ are continuous (if there is such a topology, it is unique). The corresponding results with the substitution of $H^2_{\mathrm{loc}}(G, T)$ for $H^2(G, T)$ hold for a local representation group. The point is that $H^2_{\mathrm{loc}}(G, T)$ is closed in $H^2(G, T)$ and therefore, if there is a locally compact representation group, the relative topology in $H^2_{\mathrm{loc}}(G, T)$ endows it with a topology such that all the maps $W(g, h): H^2_{\mathrm{loc}}(G, T) \to T$ are continuous and there is also then a local representation group for G. Moreover, if G and $H^2(G, T)$ are Lie groups then the representation group is also a Lie group. In this case the closed subgroup $H^2_{\mathrm{loc}}(G, T)$ will be a Lie group and therefore the local representation group is a Lie group too.

5. Examples

5.1. Factor systems of locally operating realisations

5.1.1. One of the most important symmetry groups is the three-dimensional proper rotation group $SO_3(\mathbb{R})$ i.e. the set of orthogonal transformations in three dimensions with determinant equal to +1. It acts transitively on the spheres centred at the origin. The isotopy group of a point in such a (non-trivial) sphere is isomorphic to $SO_2(\mathbb{R})$, rotations around the axis of the sphere determined by the point.

The second cohomology group of connected and compact Lie groups is easily found according to a well known result by Moore (1964, proposition 2.1): it is isomorphic to the dual group of the torsion subgroup of the first homotopy group of G, $H^2(G, T) \approx \widehat{t(\pi_1(G))}$. Moreover, $H^2(G, T)$ is finite. In the two cases with which we are concerned, $\pi_1(SO_2(\mathbb{R})) = \mathbb{Z}$ and $\pi_1(SO_3(\mathbb{R})) = \mathbb{Z}_2$ (see e.g. Boya *et al* 1978). Since $H^2(SO_3(\mathbb{R}), T) = \mathbb{Z}_2$ and $H^2(SO_2(\mathbb{R}), T) = 1$ we can conclude that any factor system of $SO_3(\mathbb{R})$ arises in some locally operating representation of $SO_3(\mathbb{R})$, this fact giving rise to the possibility of existence of half-odd spin particles. Therefore $H^2_{\mathrm{loc}}(SO_3(\mathbb{R}), T) = H^2(SO_3(\mathbb{R}), T) = \mathbb{Z}_2$. The elements of this group will be denoted $[l]$ with $l = \pm 1$. A cocycle lifting $[-1]$ will be denoted ω_{-1}.

5.1.2. A similar result holds in the case of the $(1+1)$-dimensional space-time Galilei group $G_{1,1}$ where the isotopy group is made up by boosts, i.e. Γ is isomorphic to $\mathbb{R}$. On the other side it is well known that $H^2(G_{1,1}, T) \approx \mathbb{R}^2$ (see e.g. Lévy-Leblond 1972 p 240 and 1974 p 111). The second cohomology group $H^2(\Gamma, T)$ being trivial, we have $H^2_{\mathrm{loc}}(G_{1,1}, T) = H^2(G_{1,1}, T) \approx \mathbb{R}^2$. On the contrary, for the case of the $(2+1)$-dimensional space–time Galilei group $G_{2,1}$, $H^2_{\mathrm{loc}}(G_{2,1}, T)$ is a proper subgroup of $H^2(G_{2,1}, T)$. Actually $H^2(G_{2,1}, T)$ is isomorphic to $\mathbb{R}^2$ (see e.g. Lévy-Leblond 1972 p 240). The (equivalence classes of) extensions of $G_{2,1}$ by T are characterised by a pair of real numbers $[k, m]$ indicating the two new non-vanishing commutation relations $[K_1, K_2] = kI$, $[K_i, P_j] = m\delta_{ij}$. The isotopy group Γ is the Euclidean group E(2) in two dimensions which is generated by J, K_1 and K_2. The second cohomology group $H^2(\Gamma, T)$ is $\mathbb{R}$, the corresponding classes of extensions of Γ by T being characterised by the value of the parameter k arising in the commutation relation $[K_1, K_2] = kI$. The factor system $[k, m]$ of $G_{2,1}$ becomes the factor system $[k]$ of Γ when restricted to Γ and it implies that only $[0, m]$ factor systems of $G_{2,1}$ can appear in locally operating multiplier representations, i.e. $H^2_{\mathrm{loc}}(G_{2,1}, T) \approx \mathbb{R}$.

The second cohomology group of the Galilei group in four space-time dimensions is well known to be $H^2(G_{2,1}, T) \approx \mathbb{Z}_2 \otimes \mathbb{R}$. In this case the isotopy group Γ is isomorphic to the Euclidean group in three dimensions and therefore $H^2(\Gamma, T) \approx \mathbb{Z}_2$. The theorem given in § 3 shows that $H^2_{\mathrm{loc}}(G_{3,1}, T) = H^2(G_{3,1}, T) \approx \mathbb{Z}_2 \otimes \mathbb{R}$.

5.1.3. Kinematic groups of constant and uniform Galilean electromagnetic fields were investigated by Bacry *et al* (1970) (see also the more recent papers by Beckers and Hussin (1983a, b). The electric and magnetic fields become $\boldsymbol{E}' = \boldsymbol{E} - \boldsymbol{v} \wedge \boldsymbol{B} + \boldsymbol{\theta} \cdot \boldsymbol{E}$, $\boldsymbol{B}' = \boldsymbol{B} + \boldsymbol{\theta} \wedge \boldsymbol{B}$, under infinitesimal transformation $(\boldsymbol{\theta}, \boldsymbol{v})$ of the homogeneous Galilei group. There are three different cases. The first one corresponds to non-vanishing values of the invariant $\boldsymbol{E} \cdot \boldsymbol{B}$. In this case there exists an observer for whom $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel, say in the z-direction. The second cohomology group of this group G was shown to be isomorphic with $\mathbb{R}^3$ by Hoogland (1978a) and as the isotopy group Γ is generated by the commuting elements J_3 and K_3, namely, $\Gamma = \mathbb{R}\otimes \mathrm{SO}(2)$, the second cohomology group of Γ is trivial and henceforth $H^2_{\mathrm{loc}}(G, T) = H^2(G, T) \approx \mathbb{R}^3$.

The second case is when $\boldsymbol{E}\cdot \boldsymbol{B} = 0$, but $\boldsymbol{B} \neq 0$. There exist then observers for whom $\boldsymbol{E}$ is zero and the results of the latter example still hold. The third case corresponds to pure electric uniform field which is an intrinsic notion in Galilean electromagnetism because $|\boldsymbol{B}|$ is invariant. The symmetry group is the eight-dimensional subgroup of the Galilei group generated by $\{J_3, \boldsymbol{K}, \boldsymbol{P}, P^0\}$ and the isotopy group, generated by $\{J_3, \boldsymbol{K}\}$, is isomorphic to $E(2)\otimes\mathbb{R}$. The respective second cohomology groups are $H^2(G, T)\approx \mathbb{R}^4$ and $H^2(\Gamma, T)\approx \mathbb{R}$ and therefore $H^2_{\mathrm{loc}}(G, T)\approx\mathbb{R}^3$ but $H(G, T)\approx\mathbb{R}^4$ such as indicated in Hoogland (1977, p 126).

5.2. Local representation groups

The first examples we are going to present satisfy the condition $H^2_{\mathrm{loc}}(G, T) = H^2(G, T)$ but they are chosen because they are well known for most physicists.

The first example is that of $\mathrm{SO}_3(\mathbb{R})$ acting transitively on a sphere centred at the origin. The dual group of $H^2(\mathrm{SO}_3(\mathbb{R}), T)$ is isomorphic to the cyclic group C_2 of two elements. The element generating such a group is λ defined by $\lambda([l]) = l$. Let us choose the identity automorphism on $H^2(G, T)$ and a homomorphic section s determined by a lifting $\omega_1 = s(-1)$. The maps $W_{id,s}(g, h)$ are continuous if the discrete topology on $H^2(G, T)$ is considered and therefore there exists a representation group for $\mathrm{SO}_3(\mathbb{R})$ (Santander 1977) which is also a local representation group, namely the middle group of the non-trivial central extension defined by the factor system

$$W(R', R) = \begin{cases} 1 & \text{if } \omega_{-1}(R', R) = 1 \\ \lambda & \text{if } \omega_{-1}(R', R) = -1 \end{cases}$$

which is isomorphic to the group $\mathrm{SU}_2(\mathbb{C})$. Indeed, it is a well known fact that $\mathrm{SU}_2(\mathbb{C})$ is a local representation group for $\mathrm{SO}_3(\mathbb{R})$.

As a second example we consider the above mentioned group $E(2)$ whose locally operating multiplier representations were studied by Hoogland (1978b) to prove the relevance of gauge equivalence versus the usual notion of equivalence of representations. The natural action of $E(2)$ on $\mathbb{R}^2$ is considered and the isotopy group will therefore be $\mathrm{SO}_2(\mathbb{R})$ whose second cohomology group is trivial. On the other side, $H^2(E(2), T)$ is well known to be isomorphic with $\mathbb{R}$. A class of extensions of $E(2)$ is parametrised by $[\beta]$ and its algebra is given by the commutation relations:

$$[J, P_1] = P_2 \qquad [J, P_2] = -P_1 \qquad [P_1, P_2] = \beta I$$

and a cocycle lifting $[\beta]$ is $\omega_\beta((\boldsymbol{a}', \phi'), (\boldsymbol{a}, \phi)) = \exp\{\mathrm{i}\tfrac{1}{2}\beta(\boldsymbol{a}'\wedge \boldsymbol{a}^{\phi'})_3\}$.

The second cohomology group of $\mathrm{SO}_2(\mathbb{R})$ being trivial, $H^2_{\mathrm{loc}}(E(2), T)$ coincides with $H^2(E(2), T)$. The (local) representation group can be obtained by means of the homomorphic section $s: H^2(E(2), T)\to Z^2(E(2), T)$ given by

$$s([\beta])(\boldsymbol{a}', \phi'), (\boldsymbol{a}, \phi)) = \exp\{\mathrm{i}\tfrac{1}{2}\beta(\boldsymbol{a}'\wedge \boldsymbol{a}^{\phi'})_3\}.$$

If the identity isomorphism on $H^2(E(2), T)$ is chosen and the usual topology on $\mathbb{R}$ is considered, the maps $W_{id,s}$ defined as follows

$$W_{id,s}((\boldsymbol{a}', \phi), (\boldsymbol{a}, \phi))[\beta] = \exp\{\mathrm{i}\tfrac{1}{2}\beta(\boldsymbol{a}'\wedge \boldsymbol{a}^{\phi'})_3\}$$

are continuous and $W_{id,s}\in Z^2(E(2), \widehat{H^2(E(2), T)})$ defines a central extension of $E(2)$

by $\widehat{H^2(E(2)}, T)$ which is a (local) representation group for $E(2)$. If the elements of $\widehat{H^2(E(2)}, T)$ are denoted α and they are defined as $\alpha([\beta]) = e^{i\alpha\beta}$, the local representation group is a Lie group $\overline{E(2)}$ (because $H^2(E(2), T)$ is a Lie group) with elements $(\alpha, \boldsymbol{a}, \phi)$ and composition law:

$$(\alpha', \boldsymbol{a}', \phi')(\alpha, \boldsymbol{a}, \phi) = (\alpha' + \alpha + \tfrac{1}{2}(\boldsymbol{a}' \wedge \boldsymbol{a}^{\phi'})_3, \boldsymbol{a}' + \boldsymbol{a}^{\phi'}, \phi' + \phi).$$

All the locally operating multiplier representations of $E(2)$ can be obtained from the (locally operating) linear representations of $\overline{E(2)}$ according to the methods developed in Cariñena and Santander (1979) and Cariñena *et al* (1982).

Another interesting example is the Newton–Hooke group (Bacry and Lévy-Leblond 1968, Derome and Dubois 1972), a relativistic generalisation of which was recently proposed as a dynamical group for hadrons (Roman and Haavisto 1981). For the sake of simplicity only the $(1+1)$-dimensional space–time case will be considered. The group is then a three-dimensional Lie group $N_{\pm}$, with the only non-vanishing commutation relations in its algebra $[P, H] = \pm\tau^{-2}K$, $[K, H] = P$. The parameter τ is a constant corresponding to a characteristic time of the Hooke universe. The action of N_- on the space–time, given by (b, a, v): $(x, t)(x + v\tau \sin(t/\tau) + a\cos(t/\tau), t + b)$, is transitive and the isotopy group Γ is the one-dimensional Lie subgroup generated by K, i.e., it is isomorphic with $\mathbb{R}$, and consequently, as $H^2(\Gamma, T) = 1$, $H^2_{\text{loc}}(N_-, T)$ coincides with $H^2(N_-, T)$. Since similar results hold for the N_+ case, we will only deal with the N_- group and the minus sign will be omitted.

The second cohomology group of N is easily found to be isomorphic with $\mathbb{R}$, every class of extensions of N by T being labelled by the real number $[m]$, a representative of which has the four-dimensional Lie algebra with defining relations

$$[P, H] = -\frac{1}{\tau^2}K \qquad [K, H] = P \qquad [K, P] = mI.$$

A cocycle of the class $[m]$ is given by (Derome and Dubois 1972)

$$\omega_m(h', h) = \exp\left\{im\left[\tfrac{1}{2}\left(v'^2 - \frac{1}{\tau^2}a'^2\right)\tau \sin\frac{b}{\tau}\cos\frac{b}{\tau} + a\left(v'\cos\frac{b}{\tau} - \frac{a'}{\tau}\sin\frac{b}{\tau}\right) - v'a'\sin^2\frac{b}{\tau}\right]\right\}$$

If the identity isomorphism of $H^2(N, T)$ is considered, the homomorphic section s: $H^2(N, T) \to Z^2(N, T)$ defined by $s([m])(h', h) = \omega_m(h', h)$ allows the construction of $W_{id,s}$: $N \times N \to H^2(N, T)$ by means of

$$W_{id,s}((h', h))[m] = \omega_m(h', h)$$

and the cocycle $W_{id,s} \in Z^2(N, \widehat{H^2(N, T)})$ defines a central extension of N by $\widehat{H^2(N, T)}$ which is a representation group for N. It is a four-dimensional Lie group $\bar{N}$ with elements $(\alpha, h) \equiv (\alpha, b, a, v)$ and composition law

$$(\alpha', b', a', v')(\alpha, b, a, v) = (\alpha' + \alpha + \omega_1(h', h), h'h).$$

The set of the (locally operating) multiplier representations of N can be found from that of (locally operating) linear representations of N, which are easily found if one uses the factorisation of $\bar{N}$ as a semidirect product $\bar{N} = (\bar{\mathcal{S}} \odot \mathcal{V}) \odot \mathcal{T}$ with $\bar{\mathcal{S}} = \mathbb{R} \otimes \mathcal{S}$. Here $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{V}$ denote the one-parameter subgroups of space translations, time translations and boosts respectively. This group can be shown to be isomorphic with $E(2)$, the corresponding isotopy groups being, however, non-isomorphic.

As a final example we will deal with the case of the Galilei group in $(2+1)$ space-time dimensions for which $H^2_{\rm loc}(G_{2,1}, T)$ is a proper subgroup of $H^2(G_{2,1}, T)$ such as indicated formerly. The only difference with the method explained above is the choice of a homomorphic section $s: H^2_{\rm loc}(G_{2,1}, T) \to Z^2_{\rm loc}(G_{2,1}, T)$ instead of $s: H^2(G_{2,1}, T) \to Z^2(G_{2,1}, T)$. The elements of $H^2_{\rm loc}(G_{2,1}, T)$ are parametrised by $[m]$ and the homomorphic section s given by

$$s([m])(g', g) = \omega_m(g', g) = \exp[\mathrm{i}m\tfrac{1}{2}(bv' + \boldsymbol{v}' \cdot \boldsymbol{a}^{\phi'})]$$

allows the construction of a local representation group $\bar{G}_{2,1}$ defined as the middle group of the central extension of $G_{2,1}$ by $\widetilde{H^2(G_{2,1}, T)}$ determined by the following factor system $W \in Z^2(G_{2,1}, \widetilde{H^2(G_{2,1}, T)})$:

$$W(g', g)[m] = \omega_m(g', g) = \exp[\mathrm{i}m(\tfrac{1}{2}bv'^2 + \boldsymbol{v}'\boldsymbol{a}^{\phi'})].$$

The group $\bar{G}_{2,1}$ is a seven-dimensional Lie group $\bar{G}_{2,1}$ with elements $(\alpha, g) = (\alpha, b, \boldsymbol{a}, \boldsymbol{v}, R)$ and composition law

$$(\alpha', g')(\alpha, g) = (\alpha' + \alpha + \omega_1(g', g), g'g).$$

The locally operating representations of $\bar{G}_{2,1}$ can be used to determine the locally operating multiplier representation of $G_{2,1}$, following the method of Cariñena *et al* (1982).

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